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'LETTERS' SECTION

The Zeeman–Order Topology on Minkowski Space

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Abstract

The order topology on Minkowski Space generated by Zeeman's causal order is an elementary example of a topology whose autohomeomorphism group is a Poincaré dilation group.

Recently interest was revived (Nanda, 1971) in Zeeman's (1967) idea of topologising Minkowski Space-Time with a topology depending on the local cone structure. Originally Zeeman proposed his 'fine topology' (along with several others) as a topology probably more physically significant than the usual $\mathbf{R}^4$ topology (arising from the 4-manifold structure with a global chart). We briefly describe here an elementary topology induced on Minkowski space by Zeeman's causal order (1964). No mention of it is to be found in the literature probably because it is not finer than $\mathbf{R}^4$. It does, however, have some interesting features.

Let M be Minkowski space and $(\langle)$ be Zeeman's causal order. For any event $x \in M$ define

$$\langle x| \equiv \{y \in M \vdash y < x \text{ or } y = x\}, \quad |x\rangle \equiv \{y \in M \vdash y > x \text{ or } y = x\}.$$

Then the families

$$\mathscr{B}_L \equiv \{\langle x| \vdash x \in M\}, \quad \mathscr{B}_R \equiv \{|x\rangle \vdash x \in M\}.$$

form bases for topologies τ_L, τ_R respectively on M. Let M_L, M_R denote M topologised with either τ_L, τ_R, the left, right order topologies on M. Now it is clear that the space-time reversal $T\colon M \to M$, $T\colon x \mapsto -x \; \forall \; x \in M$ is a homeomorphism $T\colon M_R \cong M_L$ so we need consider on M_L.

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It is easy to show that M_L is rather complicated topologically. In short, M_L is T_0 but not T_1 and thus not T_2. Neither is M_L T_3 and hence it is not T_4. The poor separability is a measure of the coarseness of the topology. M_L is connected and locally connected, first category and first countable, whilst it is neither compact nor locally compact. (M_L is not compact because it is not paracompact.)

On the other hand, M_L does have some of the properties suggested by Zeeman to confer physical significance (when taken together with the property of being finer than $\mathbf{R}^4$). τ_L induces discrete topologies on light rays and on any space-like subspace of M. ($A \subset M$ is space-like iff $\forall x, y \in A$ $x \nless y$ and $y \nless x$.) Of more interest is the fact that the autohomeomorphism group of M_L is the orthochronous Poincaré-dilation group $D\uparrow \cong P\uparrow \; Xp \; \mathbf{R}^+$. (Here we mean a semi-direct product of the Poincaré group $P\uparrow$ by the multiplicative group $\mathbf{R}^+$ of positive reals with $p \in \text{Hom}(\mathbf{R}^+, \text{Aut}(P\uparrow))$ given by:

$$p: \lambda \mapsto p(\lambda): (x, \Lambda) \mapsto (\lambda x, \Lambda), \lambda \in \mathbf{R}^+, (x, \Lambda) \in P\uparrow.)$$

The latter assertion is demonstrated using the fact that the autohomeomorphism group of M_L can be shown to be the group of all bijections of M onto itself which bipreserve:

$$(x < y \Rightarrow f(x) < f(y) \quad \text{and} \quad f^{-1}(x) < f^{-1}(y))$$

the Zeeman order, and the theorem of Zeeman (1964) stating that the group of all such bijections is the group $D\uparrow$. The fact that the autohomeomorphism group of Zeeman's fine topology is the full Poincaré dilatation group is probably its most attractive feature.

References

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